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On Addition Formulae of KP, mKP and BKP hierarchies

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1 The addition formula for the τ-function of the KP hierarchy

Let

$$[\alpha]=(\alpha,\frac{\alpha^{2}}{2},\frac{\alpha^{3}}{3},\ldots),\qquad \xi(t,\lambda)=\sum_{n=1}^{\infty}t_{n}\lambda^{n},\qquad t=(t_{1},t_{2},t_{3},\cdots).$$

The KP hierarchy is a system of equations for a function $\tau(t)$ given by

$$\oint e^{\xi(t'-t,\lambda)}\tau(t'-[\lambda^{-1}])\tau(t+[\lambda^{-1}])\frac{d\lambda}{2\pi i}=0. \tag{1}$$

Here $\oint$ means a formal algebraic operator extracting the coefficient of z^{-1} of Laurent series:

$$\oint\frac{dz}{2\pi i}\sum_{n=-\infty}^{\infty}a_{n}z^{n}=a_{-1}.$$

Set $t=x+y, t'=x-y$. Then (1) becomes

$$\oint e^{-2\xi(y,\lambda)}\tau(x-y-[\lambda^{-1}])\tau(x+y+[\lambda^{-1}])\frac{d\lambda}{2\pi i}=0. \tag{2}$$

Set

$$y=\frac{1}{2}(\sum_{i=1}^{m-1}[\beta_{i}]-\sum_{i=1}^{m+1}[\alpha_{i}]).$$

By virtue of the identity

$$\sum_{n=1}^{\infty}\frac{x^{n}}{n}=-\log(1-x),$$

the exponential factor $e^{-2\xi(y,\lambda)}$ reduces to a rational function of $\lambda, \alpha_{i}, \beta_{i}$ as

$$e^{-2\xi(y,\lambda)}=\frac{\prod_{i=1}^{m-1}(1-\beta_{i}\lambda)}{\prod_{i=1}^{m+1}(1-\alpha_{i}\lambda)}.$$

Finally shifting the variable x as

$$x\rightarrow x+\frac{1}{2}(\sum_{i=1}^{m-1}[\beta_{i}]-\sum_{i=1}^{m+1}[\alpha_{i}]),$$

we get the following addition formulae of τ-function

$$\sum_{i=1}^{m+1}(-1)^{i-1}\zeta(x;\beta_{1},\ldots,\beta_{m-1},\alpha_{i})\zeta(x;\alpha_{1},\ldots,\hat{\alpha}_{i},\ldots,\alpha_{m+1})=0,\qquad m\geq 2, \tag{3}$$

where

$$\zeta(x;\alpha_1,\dots,\alpha_n) = \Delta(\alpha_1,\dots,\alpha_n)\tau(x+[\alpha_1]+\cdots+[\alpha_n]),$$
$$\Delta(\alpha_1,\dots,\alpha_n) = \prod_{i<j}(\alpha_i-\alpha_j),$$

and, $\hat{\alpha_i}$ denotes to remove α_i.

Example 1 *In the case of $m=2$, we have*

$$\begin{aligned}&\alpha_{12}\alpha_{34}\tau(x+[\alpha_1]+[\alpha_2])\tau(x+[\alpha_3]+[\alpha_4])\\ &-\alpha_{13}\alpha_{24}\tau(x+[\alpha_1]+[\alpha_3])\tau(x+[\alpha_2]+[\alpha_4])\\ &+\alpha_{14}\alpha_{23}\tau(x+[\alpha_1]+[\alpha_4])\tau(x+[\alpha_2]+[\alpha_3])=0,\end{aligned} \tag{4}$$

where $\alpha_{ij}=\alpha_i-\alpha_j$.

We call (4) 'the three terms equation'. We have derived (4) from (1). In fact, the converse is true.

Theorem 1 *The three terms equation (4) is equivalent to the KP hierarchy (1).*

This theorem has been proved by Takasaki and Takebe [25]. They proved the theorem by constructing the wave function of the KP-hierarchy. To do it they used the differential Fay identity which is a certain limit of (4). Here we give an alternative and direct proof of the theorem. Theorem 1 is proved by using the following propositions.

Proposition 1 *The KP hierarchy (1) is equivalent to (3).*

Proposition 2 *The following formula follows from (4):*

$$\frac{\tau(x+\sum_{i=1}^m[\beta_i]-\sum_{i=1}^m[\alpha_i])}{\tau(x)} = \frac{\prod_{i,j=1}^m(\beta_i-\alpha_j)}{\prod_{i<j}\alpha_{ij}\beta_{ji}}\det\left(\frac{\tau(x+[\beta_i]-[\alpha_j])}{(\beta_i-\alpha_j)\tau(x)}\right)_{1\le i,j\le m}, \quad m\ge 2. \tag{5}$$

Proposition 3 *The Plücker relations for the determinant of the right hand side of (5) give the addition formulae (3).*

Proposition 1 is proved using the properties of symmetric functions. Proposition 2 is proved by using the Sylvester's theorem on determinants.

2 The mKP hierarchy

Let $\tau_l(t)$ ($l\in\mathbb{Z}$) be τ-functions of the modified KP (mKP) hierarchy. We use the same notation as that for KP hierarchy ($[\alpha]$, $\xi(t,\lambda)$, etc.).

The mKP hierarchy is given by the bilinear equation of the form

$$\oint e^{\xi(t-t',\lambda)}\lambda^{l-l'}\tau_l(t-[\lambda^{-1}])\tau_{l'}(t'+[\lambda^{-1}])\frac{d\lambda}{2\pi i}=0, \quad l\ge l'. \tag{6}$$

Set $t=x-y, t'=x+y$. Then (6) becomes

$$\oint e^{-2\xi(y,\lambda)}\lambda^{l-l'}\tau_l(x-y-[\lambda^{-1}])\tau_{l'}(x+y+[\lambda^{-1}])\frac{d\lambda}{2\pi i}=0, \quad l\ge l'. \tag{7}$$

Let $l - l' = k \geq 0$. Set

$$y = \frac{1}{2}(\sum_{i=1}^{m-2}[\beta_i] - \sum_{i=1}^{m+k}[\alpha_i]).$$

The exponential factor in (7) reduces to a rational function of $\lambda, \alpha_i, \beta_i$ as in the KP case:

$$\exp\left(-\xi(\sum_{i=1}^{m-2}[\beta_i] - \sum_{i=1}^{m+k}[\alpha_i], \lambda)\right) = \frac{\prod_{i=1}^{m-2}(1-\beta_i\lambda)}{\prod_{i=1}^{m+k}(1-\alpha_i\lambda)}.$$

Computing the integral as the KP case and shift the variable x as

$$x \to x + \frac{1}{2}(\sum_{i=1}^{m-2}[\beta_i] - \sum_{i=1}^{m+k}[\alpha_i]),$$

and we get the following addition formulae of the mKP hierarchy:

$$\begin{aligned}&\sum_{i=1}^{m+k}(-1)^{i-1}\zeta_l(x;\beta_1,\ldots,\beta_{m-2},\alpha_i)\zeta_{l+k}(\alpha_1,\ldots,\hat{\alpha_i},\ldots,\alpha_{m+k}) = 0\\ &l \in \mathbb{Z}, \quad k \geq 0, \quad m \geq 2,\end{aligned} \tag{8}$$

where

$$\zeta_(x;\alpha_1,\ldots,\alpha_n) = \Delta(\alpha_1,\ldots,\alpha_n)\tau_l(x + \sum_{i=1}^{n}[\alpha_i]).$$

Example 2 *The case $l - l' = 1$ and $m = 2$ of (8) is*

$$\begin{aligned}&\alpha_{23}\tau_l(x+[\alpha_1])\tau_{l+1}(x+[\alpha_2]+[\alpha_3])\\ &-\alpha_{13}\tau_l(x+[\alpha_2])\tau_{l+1}(x+[\alpha_1]+[\alpha_3])\\ &+\alpha_{12}\tau_l(x+[\alpha_3])\tau_{l+1}(x+[\alpha_1]+[\alpha_2]) = 0.\end{aligned} \tag{9}$$

We call this equation (9) 'the three terms equation of the mKP hierarchy'.
In this case, we have

Theorem 2 *The three terms equation (9) is equivalent to the mKP hierarchy (6).*

Theorem 2 has been proved by Takebe. We give another and direct proof of it. Similarly to the case of the KP hierarchy, this theorem is proved by using the following propositions.

Proposition 4 *The mKP hierarchy (6) is equivalent to (8).*

Proposition 5 *The following equation follows from (9):*

$$\begin{aligned}&\frac{\tau_{l+1}(x+\sum_{i=1}^{n}[\alpha_i] - \sum_{i=1}^{n-1}[\beta_i])}{\tau(x)}\\ &= C\det\begin{pmatrix} \frac{\tau_l(x+[\alpha_1]-[\beta_1])}{(\alpha_1-\beta_1)\tau_l(x)} & \cdots & \frac{\tau_l(x+[\alpha_1]-[\beta_{n-1}])}{(\alpha_1-\beta_{n-1})\tau_l(x)} & \frac{\tau_{l+1}(t+[\alpha_1])}{\tau_l(x)} \\ \frac{\tau_l(x+[\alpha_2]-[\beta_1])}{(\alpha_2-\beta_1)\tau_l(x)} & \cdots & \frac{\tau_l(x+[\alpha_2]-[\beta_{n-1}])}{(\alpha_2-\beta_{n-1})\tau_l(x)} & \frac{\tau_{l+1}(t+[\alpha_2])}{\tau_l(x)} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\tau_l(x+[\alpha_n]-[\beta_1])}{(\alpha_n-\beta_1)\tau_l(x)} & \cdots & \frac{\tau_l(x+[\alpha_n]-[\beta_{n-1}])}{(\alpha_n-\beta_{n-1})\tau_l(x)} & \frac{\tau_{l+1}(t+[\alpha_n])}{\tau_l(x)} \end{pmatrix},\end{aligned} \tag{10}$$

where

$$C = \frac{\prod_{i=1}^{n}\prod_{j=1}^{n-1}(\alpha_i - \beta_j)}{(\prod_{i<j}^{n-1}\beta_{ij})(\prod_{i>j}^{n}\alpha_{ij})}.$$

Proposition 6 *The Plücher relations for the determinant of right hand side of (10) gives (8) with* $k = 1$.

Lemma 1 *Equation (8) follows from (9).*

Using free fermions, we can derive the equation (10).
Following [1] let $\psi_n,\ \psi_n^*$ be free fermionic operators with the following anticommutation relations:

$$[\psi_n, \psi_m]_+ = [\psi_n^*, \psi_m^*]_+ = 0,\ \ [\psi_n, \psi_m^*]_+ = \delta_{mn}.$$

They generate an infinite dimensional Clifford algebra. We define the generating functions of free fermions as

$$\psi(\lambda) = \sum_{i=1}^{\infty}\psi_i\lambda^i,\ \ \psi^*(\lambda) = \sum_{i=1}^{\infty}\psi_i^*\lambda^{-i}.$$

For $n \in \mathbb{Z}$, set

$$H(x) = \sum_{n=1}^{\infty}x_nH_n,\ \ H_n = \sum_{i\in\mathbb{Z}} :\psi_i\psi_{i+n}^*: .$$

Then we introduce a vacuum $|0\rangle$ and the dual vacuum $\langle 0|$. These vacuum have the following properties:

$$\begin{aligned}&\psi_n|0\rangle = 0,\ (n<0),\ \ \psi_n^*|0\rangle = 0,\ (n \geq 0)\\ &\langle 0|\psi_n = 0,\ (n \geq 0),\ \ \langle 0|\psi_n^* = 0,\ (n<0)\end{aligned}$$

We need the shifted vacua $|l\rangle$ and the dual vacua $\langle l|$ defined by

$$|l\rangle = \begin{cases}\psi_{l-1}\cdots\psi_0|0\rangle,\ n>0\\ \psi_l^*\cdots\psi_{-1}^*|0\rangle,\ n<0\end{cases}$$
$$\langle l| = \begin{cases}\langle 0|\psi_0^*\cdots\psi_{n-1}^*,\ n>0\\ \langle 0|\psi_{-1}\cdots\psi_n,\ n<0.\end{cases}$$

It is easy to check the following properties:

$$\begin{aligned}&\psi_n|l\rangle = 0,\ n<l,\ \ \psi_n^*|l\rangle = 0,\ n \geq l\\ &\langle l|\psi_n = 0,\ n \geq l,\ \ \langle l|\psi_n^* = 0,\ n<l.\end{aligned}$$

Proposition 7 *We get the equation (10) by the following equation:*

$$\begin{aligned}&\frac{\langle l|\psi^*(\alpha_1^{-1})\cdots\psi^*(\alpha_n^{-1})\psi(\beta_{n-1}^{-1})\cdots\psi(\beta_1^{-1})e^{H(x)}g|l+1\rangle}{\langle l|e^{H(x)}g|l\rangle}\\ &= (-1)^{n-1}\det\begin{pmatrix}a_{11} & \cdots & a_{1,n-1} & b_1\\ \vdots & \ddots & \vdots & \vdots\\ a_{n1} & \cdots & a_{n,n-1} & b_n\end{pmatrix},\end{aligned} \tag{11}$$

where

$$a_{ij} = \frac{\langle l|\psi^*(\alpha_i^{-1})\psi(\beta_j^{-1})e^{H(x)}g|l\rangle}{\langle l|e^{H(x)}g|l\rangle},$$
$$b_i = \frac{\langle l|\psi^*(\alpha_i^{-1})e^{H(x)}g|l+1\rangle}{\langle l|e^{H(x)}g|l\rangle},$$

and

$$G = \{g \in A|\exists g^{-1},\, gVg^{-1} = V,\, gV^*g^{-1} = V^*\},\ \ V = \oplus_{i\in\mathbf{Z}}\mathbb{C}\psi_i,\ V^* = \oplus_{i\in\mathbf{Z}}\mathbb{C}\psi_i^*,$$

and A is the Clifford algebra.

Equation (11) can be derived by using the generalized Wick's theorem. For $l \in \mathbb{Z}$, we can get the (10) by considering

$$\tau_l(x) = \langle l|e^{H(x)}g|l\rangle,\ g \in G.$$

3 The BKP hierarchy

Let $\tau(t)$ be the τ-function of the BKP hierarchy. In this case, the time variable is $t = (t_1, t_3, t_5, \cdots)$. We set

$$[\alpha]_o = (\alpha, \frac{\alpha^3}{3}, \frac{\alpha^5}{5}, \ldots), \qquad \tilde{\xi}(t,\lambda) = \sum_{n=1}^{\infty} t_{2n-1}\lambda^{2n-1}.$$

The BKP hierarchy is defined by

$$\oint e^{\tilde{\xi}(t-t',\lambda)}\tau(t - 2[\lambda^{-1}]_o)\tau(t' + 2[\lambda^{-1}]_o)\frac{d\lambda}{2\pi i\lambda} = \tau(t)\tau(t'). \tag{12}$$

Set $t = x + y, t' = x - y$. We get

$$\oint e^{-2\tilde{\xi}(y,\lambda)}\tau(x - y - 2[\lambda^{-1}]_o)\tau(x + y + 2[\lambda^{-1}]_o)\frac{d\lambda}{2\pi i\lambda} = \tau(x+y)\tau(x-y). \tag{13}$$

Set

$$y = \sum_{i=1}^{n}[\alpha_i]_o.$$

By separating $-2\sum_{n=1}^{\infty} t_{2n-1}\lambda^{2n-1}$ as

$$-2\sum_{n=1} t_{2n-1}\lambda^{2n-1} = -\sum_{n=1}^{\infty} t_n\lambda^n + \sum_{n=1}^{\infty} t_n(-\lambda)^n,$$

we get

$$\exp\left(-2\tilde{\xi}(\sum_{i=1}^{n}[\alpha_i]_o, \lambda)\right) = \prod_{i=1}^{n}\frac{1-\alpha_i\lambda}{1+\alpha_i\lambda}.$$

Computing the integral by taking residues as before and shifting x appropriately, we have

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{\tau(x+2[\alpha_i]_o)}{\tau(x)}A^{-1}_{1\ldots\hat{i}\ldots n}\frac{\tau(x+2\sum_{l\neq i}^{n}[\alpha_l]_o)}{\tau(x)}$$
$$-A^{-1}_{1\ldots n}\frac{\tau(x+2\sum_{l=1}^{n}[\alpha_l]_o)}{\tau(x)}=0,\qquad n: odd, \tag{14}$$
$$\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\alpha_{i,n}}{\tilde{\alpha}_{i,n}}\frac{\tau(x+2[\alpha_i]_o+2[\alpha_n]_o)}{\tau(x)}A^{-1}_{1\ldots\hat{i}\ldots n-1}\frac{\tau(x+2\sum_{l\neq i}^{n}[\alpha_l]_o)}{\tau(x)}$$
$$-A^{-1}_{1\ldots n}\frac{\tau(x+2\sum_{l=1}^{n}[\alpha_l]_o)}{\tau(x)}=0,\qquad n: even. \tag{15}$$

Here $A_{1\ldots n}$ is defined by

$$A_{1\ldots n}=\prod_{1=i<j}^{n}\frac{\tilde{\alpha}_{ij}}{\alpha_{ij}},\qquad \tilde{\alpha}_{ij}=\alpha_i+\alpha_j,\qquad \alpha_{ij}=\alpha_i-\alpha_j.$$

Example 3 *The case $n=3$ of (14) is*

$$\begin{aligned}\frac{\tau(x+2\sum_{i=1}^{3}[\alpha_i]_o)}{\tau(x)}=A_{123}\Big(&\frac{\tau(x+2[\alpha_1]_o)}{\tau(x)}\frac{\alpha_{23}}{\tilde{\alpha}_{23}}\frac{\tau(x+2[\alpha_2]_o+2[\alpha_3]_o)}{\tau(x)}\\&-\frac{\tau(x+2[\alpha_2]_o)}{\tau(x)}\frac{\alpha_{13}}{\tilde{\alpha}_{13}}\frac{\tau(x+2[\alpha_1]_o+2[\alpha_3]_o)}{\tau(x)}\\&+\frac{\tau(x+2[\alpha_3]_o)}{\tau(x)}\frac{\alpha_{12}}{\tilde{\alpha}_{12}}\frac{\tau(x+2[\alpha_1]_o+2[\alpha_2]_o)}{\tau(x)}\Big).\end{aligned} \tag{16}$$

We call Equation (16) 'the four terms equation of the BKP hierarchy'.

Example 4 *The case of $n=4$ of (15) is*

$$\begin{aligned}\frac{\tau(x+2\sum_{i=1}^{4}[\alpha_i]_o)}{\tau(x)}=A_{1234}\Big(&\frac{\alpha_{14}}{\tilde{\alpha}_{14}}\frac{\tau(x+2[\alpha_1]_o+2[\alpha_4]_o)}{\tau(x)}\frac{\alpha_{23}}{\tilde{\alpha}_{23}}\frac{\tau(x+2[\alpha_2]_o+2[\alpha_3]_o)}{\tau(x)}\\&-\frac{\alpha_{24}}{\tilde{\alpha}_{24}}\frac{\tau(x+2[\alpha_2]_o+2[\alpha_4]_o)}{\tau(x)}\frac{\alpha_{13}}{\tilde{\alpha}_{13}}\frac{\tau(x+2[\alpha_1]_o+2[\alpha_3]_o)}{\tau(x)}\\&+\frac{\alpha_{34}}{\tilde{\alpha}_{34}}\frac{\tau(x+2[\alpha_3]_o+2[\alpha_4]_o)}{\tau(x)}\frac{\alpha_{12}}{\tilde{\alpha}_{12}}\frac{\tau(x+2[\alpha_1]_o+2[\alpha_2]_o)}{\tau(x)}\Big).\end{aligned} \tag{17}$$

Equation (17) of example 4 can be derived from Equation (16).
Then,

Theorem 3 *The four terms equation (16) is equivalent to the bilinear identity of the BKP hierarchy (12).*

Theorem 3 is proved by Takasaki [23]. Here we give an alternative and direct proof of it.
In order to explain the strategy, we introduce the Pfaffian. Set A= $(a_{ij})_{1\leq i,j\leq 2m}$ is a skew-symmetric matrix with the degree $2m$. Then, the Pfaffian is defined by

$$\det A=(PfA)^2,\qquad PfA=a_{12}a_{34}\cdots a_{2m-1,2m}-\cdots.$$

Following [8] we denote PfA by $(1,2,3,\ldots,2m)$:

$$PfA=(1,2,3,\ldots,2m).$$

It is directly defined by

$$(1,2,3,\ldots,2m) = \sum sgn(i_1,\ldots,i_{2m}) \cdot (i_1,i_2)(i_3,i_4)\cdots(i_{2m-1},i_{2m}), \qquad (i.j) = a_{ij},$$

where the sum is over all permutations of (1,...,2m) such that

$$i_1 < i_3 < \cdots < i_{2m-1}, \quad i_1 < i_2, \cdots, i_{2m-1} < i_{2m},$$

and $sgn(i_1,\ldots,i_{2m})$ is the signature of the permutations $(i_1,\ldots,i_{2m})$.
The Pfaffian can be expanded as

$$(1,2,3,\ldots,2m) = \sum_{j=2}^{2m}(-1)^j(1,j)(2,3,\ldots,\hat{j},\ldots,2m).$$

For example, in the case of m=2,

$$(1,2,3,4) = (1,2)(3,4) - (1,3)(2,4) + (1,4)(2,3).$$

Let us define the components of Pfaffian by

$$(0,j) = \frac{\tau(x+2[\alpha_j]_o)}{\tau(x)} \quad , \qquad (i,j) = \frac{\alpha_{ij}}{\tilde{\alpha}_{ij}}\frac{\tau(x+2[\alpha_i]_o+2[\alpha_j]_o)}{\tau(x)}.$$

Then, we rewrite (16) and (17) as

$$\frac{\tau(x+2\sum_{i=1}^{3}[\alpha_i]_o)}{\tau(x)} = A_{123}(0,1,2,3), \tag{18}$$

$$\frac{\tau(x+2\sum_{i=1}^{4}[\alpha_i]_o)}{\tau(x)} = A_{1234}(1,2,3,4). \tag{19}$$

Theorem 3 can be proved similarly to the KP case using the following propositions.

Proposition 8 *The BKP hierarchy (12) is equivalent to (14) and (15).*

Proposition 9 *The following equations follow from (16):*

$$\frac{\tau(x+2\sum_{i=1}^{n}[\alpha_i]_o)}{\tau(x)} = A_{1\ldots n}(0,1,2,\ldots,n), \qquad n: odd, \tag{20}$$

$$\frac{\tau(x+2\sum_{i=1}^{n}[\alpha_i]_o)}{\tau(x)} = A_{1\ldots n}(1,2,\ldots,n), \qquad n: even. \tag{21}$$

There exists an analogue of the Plücker relations for Pfaffians [18].
Then we have

Proposition 10 *The Plücker relation for the Pfaffians of the right hand side of (20) and (21) give the addition formulae (14) and (15) respectively.*

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